Consider ϕ^4- theory with Lagrangian:

$$\mathcal{L}_{\phi^4} = \frac{1}{2}\mu^2\phi^2 + \frac{\lambda}{4!}\phi^4$$

Last time:

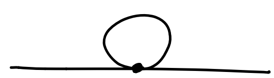
$$\Gamma_R^{(2)}(0, m^2, g) = m^2,$$

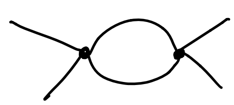
$$\frac{\partial}{\partial K^2}\Gamma_R^{(2)}(K, m^2, g)\Big|_{K^2=0} = 1$$

$$\Gamma_R^{(4)}(K_i, m^2, g)\Big|_{K_i=0} = g$$

<u>Derivation:</u>

a) At one-loop:

$\Gamma^{(2)} =$ 

$\Gamma^{(4)} =$ 

From the first graph we get

$$\mu^2 = m_1^2 - \frac{\lambda}{2}\int^{\Lambda} \frac{1}{q^2+\mu^2}$$

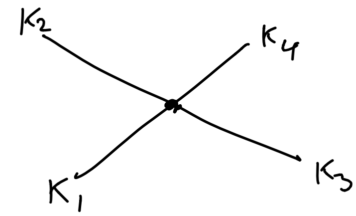
where we take m_1 to be finite

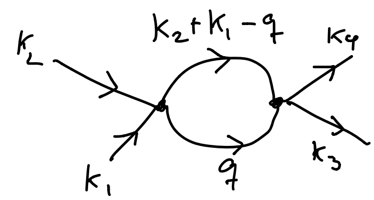
$\longrightarrow$ rewrite as:

$$\mu^2 = m_1^2 - \frac{\lambda}{2}\int^{\Lambda} \frac{1}{q^2+m_1^2+\mathcal{O}(\lambda)}$$

$$= m_1^2 - \frac{\lambda}{2}\int^{\Lambda} \frac{1}{q^2+m_1^2} + \mathcal{O}(\lambda^2) \qquad (1)$$

The 4-point function up to one-loop contains





+ 2 permutations

$$\longrightarrow \Gamma^{(4)}(k_i) = \lambda - \frac{\lambda^2}{2}\int \frac{1}{(q^2+\mu^2)\left[(k_1+k_2-q)^2+\mu^2\right]} \qquad (2)$$

+ 2 permutations

$\longrightarrow$ has ultraviolet logarithmic divergence for $d \to 4$

No other vertex function has a UV divergence! (Recall $\delta = -n\delta_r + (d + E - \frac{1}{2}Ed)$, so for $\delta_4 = 0$ and $d = E = 4$, we get $\delta = 0$, for $E > 4$ δ becomes negative and hence finite)

In (2) μ^2 can be replaced by m_1^2 and the difference is higher order:

$$\Gamma^{(4)}(k_i) = \lambda - \frac{\lambda^2}{2}\int \frac{1}{(q^2+\mu^2)\left[(k_1+k_2-q)^2+m_1^2\right]} + O(\lambda^3)$$

Define "renormalized coupling constant":

$$\frac{g_1}{4!} = \frac{\lambda}{4!} - \frac{\lambda^2}{16}\int \frac{1}{(q^2+m_1^2)^2}$$

Using eq. (2), we can rewrite this as

$$\mu^2 = m_1^2 + \frac{g_1}{2}\int \frac{1}{q^2+m_1^2} + \mathcal{O}(g_1^2)$$
$$\lambda = g_1 + \frac{3}{2}g_1^2 \int \frac{1}{(q^2+m_1^2)^2} + \mathcal{O}(g_1^3) \qquad (3)$$

$\longrightarrow$ finite m_1^2 and g_1 imply infinite "bare" parameters μ^2 and λ in $d=4$.

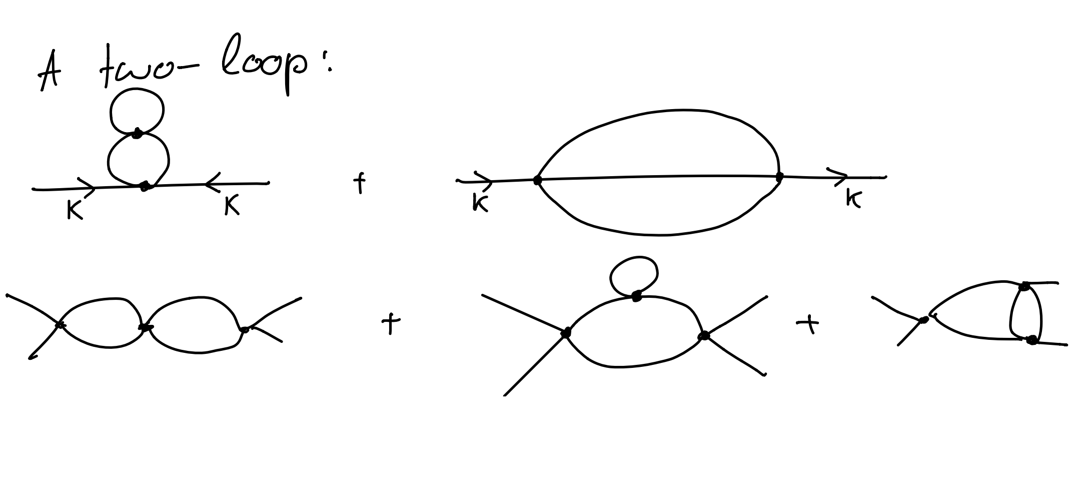
Equation (2) now becomes

$$\Gamma^{(4)}(K_i) = g_1 - \frac{g_1^2}{2}\int\left[\frac{1}{(q^2+m_1^2)[(K_1+K_2-q)^2+m_1^2]} - \frac{1}{(q^2+m_1^2)^2}\right]$$
$$+ \text{ 2 permutations } + \mathcal{O}(g_1^3)$$

We see that $\Gamma^{(4)}(0) = g_1$ and $\Gamma^{(4)}(K_i)$ is finite at $d=4$ (divergences in Λ cancel in the difference, exercise)

$\longrightarrow$ choice of m_1^2 and g_1 is special: fixed at <u>zero</u> external momenta

b) A two-loop:



Now the renormalized mass withe prescription $m_1^2 = \Gamma^{(2)}(k=0)$ becomes

$$m_1^2 = \mu^2 + \frac{\lambda}{2} D_1(\mu^2,\Lambda) - \frac{\lambda^2}{4} D_2(\mu^2,\Lambda) D_1(\mu^2,\Lambda) - \frac{\lambda^2}{6} D_3(0,\mu^2,\Lambda) \qquad (4)$$

where

$$D_1(\mu^2,\Lambda) = \int^{\Lambda} \frac{1}{q^2+\mu^2}$$

$$D_2(\mu^2,\Lambda) = \int^{\Lambda} \frac{1}{(q^2+\mu^2)^2}$$

$$D_3(k,\mu^2,\Lambda) = \int^{\Lambda} \frac{1}{(q_1^2+\mu^2)(q_2^2+\mu^2)\left[(k-q_1-q_2)^2+\mu^2\right]}$$

Last two terms are of order 2 loops

$\longrightarrow$ μ^2 can be replaced by m_1^2 in these

Rewrite D_1 as :

$$D_1(\mu^2,\Lambda) = \int \frac{1}{q^2+m_1^2-\frac{\lambda}{2}D_1(m_1^2,\Lambda)}$$

$$= D_1(m_1^2,\Lambda) + \frac{\lambda}{2} D_2(m_1^2,\Lambda) D_1(m_1^2,\Lambda)$$

So eq. (4) becomes

$$\mu^2 = m_1^2 - \frac{\lambda}{2} D_1(m_1^2,\Lambda) + \frac{\lambda^2}{6} D_3(0,m_1^2,\Lambda)$$

$$\longrightarrow \Gamma^{(2)}(k) = k^2 + m_1^2 - \frac{\lambda^2}{6}\left[D_3(k,m_1^2,\Lambda) - D_3(0,m_1^2,\Lambda)\right] \qquad (5)$$

$\longrightarrow$ has logarithmic divergence (exercise) in $d=4$

Nowe come to $\Gamma^{(4)}(k_i)$:

$$\Gamma^{(4)}(k_i) = \lambda - \frac{\lambda^2}{2}\left[I(k_1+k_2,\mu^2,\Lambda) + 2 \text{ permutations}\right]$$

μ^2 can be replaced by m_1^2 up to $O(\lambda^4)$

$$\left\{\begin{aligned} &+\frac{\lambda^3}{4}\left[I^2(k_1+k_2,\mu^2,\Lambda) + 2 \text{ permutations}\right] \\ &+\frac{\lambda^3}{4}\left[I_3(k_1+k_2,\mu^2,\Lambda)D_1(\mu^2,\Lambda) + 2 \text{ permutations}\right] \\ &+\frac{\lambda^3}{2}\left[I_4(k_i,\mu^2,\Lambda) + 5 \text{ permutations}\right] \end{aligned}\right. \qquad (6)$$

where

$$I(k,\mu^2,\Lambda) = \int \frac{1}{(q^2+\mu^2)[(k-q)^2+\mu^2]}$$

$$I_3(k,\mu^2,\Lambda) = \int \frac{1}{(q^2+\mu^2)^2[(k-q)^2+\mu^2]}$$

$$I_4(k_i,\mu^2,\Lambda) = \int \frac{1}{(q_1^2+\mu^2)[(k_1+k_2-q_1)^2+\mu^2](q_2^2+\mu^2)[(k_3+q_1-q_2)^2+\mu^2]}$$

After mass renormalization, the fourth term cancels and we get

$$\Gamma^{(4)}(k_i) = \lambda - \frac{\lambda^2}{2}\left[I(k_1+k_2,m_1^2,\Lambda) + 2 \text{ permutations}\right]$$
$$+\frac{\lambda^3}{4}\left[I^2(k_1+k_2,m_1^2,\Lambda) + 2 \text{ permutations}\right]$$
$$+\frac{\lambda^3}{2}\left[I_4(k_i,m_1^2,\Lambda) + 5 \text{ permutations}\right] \quad (7)$$

$\rightarrow$ logarithmically divergent

$\rightarrow$ introduce renormalized coupling constant

$$g_1 = \lambda - \frac{3}{2}\lambda^2 D_2(m_1^2,\Lambda) + \frac{3}{4}\lambda^3\left[D_2(m_1^2,\Lambda)\right]^2 + 3\lambda^3 I_4(K_i=0, m_1^2,\Lambda)$$

Inverting this gives :

$$\lambda = g_1 + \frac{3}{2}g_1^2 D_2(m_1^2,\Lambda) + \frac{15}{4}g_1^3\left[D_2(m_1^2,\Lambda)\right]^2 - 3g_1^3 I_4(K_i=0, m_1^2,\Lambda) + \mathcal{O}(g_1^4)$$

$$\rightarrow \Gamma^{(4)}(K_i) = g_1 - \frac{1}{2}g_1^2\Big(\left[I(K_1+K_2,m_1^2,\Lambda) - D_2(m_1^2,\Lambda)\right] + 2 \text{ permutations}\Big)$$
$$+ \frac{1}{4}g_1^3\Big(\left[I(K_1+K_2,m_1^2,\Lambda) - D_2(m_1^2,\Lambda)\right]^2 + 2 \text{ permutations}\Big)$$
$$+ \frac{1}{2}g_1^3\Big(\left[I_4(K_i,m_1^2,\Lambda) - I_4(0,m_1^2,\Lambda)\right] - D_2(m_1^2,\Lambda)\left[I(K_1+K_2,m_1^2,\Lambda) - D_2(m_1^2,\Lambda) + 5 \text{ perm.}\Big) \quad (8)$$

Also rewrite $\Gamma^{(2)}$ (eq. (5)) in terms of g_1 :

$$\Gamma^{(2)}(K,m_1^2,g_1) = K^2 + m_1^2 - \frac{g_1^2}{6}\left[D_3(K,m_1^2,\Lambda) - D_3(0,m_1^2,\Lambda)\right]$$

$\rightarrow$ diverges as $\ln\Lambda$ in $d=4$

$\rightarrow$ introduce new two-point vertex :

$$\Gamma_R^{(2)} = Z_\phi(g_1, m_1, \Lambda)\,\Gamma^{(2)}(K,m_1^2,\Lambda) \qquad (9)$$

where

$$Z_\phi = 1 + g_1 z_1 + g_1^2 z_2 + \cdots$$

Thus:

$$\Gamma_R^{(2)}(k, m_1^2, \Lambda) = k^2 + m_1^2(1 + g_1^2 z_2)$$
$$- \frac{1}{6} g_1^2 \left[D_3(k, m_1^2, \Lambda) - D_3(0, m_1^2, \Lambda) - 6 z_2 k^2 \right] \qquad (10a)$$

(z_1 was set to zero)

Using (exercise)

$$D_3(k, m_1^2, \Lambda) = D_3(0, m^2, \Lambda) + \Big(\underbrace{\frac{\partial}{\partial k^2} D_3(k, m_1^2, \Lambda) \Big|_{k=0}}_{\sim \ln \Lambda} \Big) k^2 + \underbrace{O(k^4)}_{\text{convergent}}$$

we see that the prescription

$$z_2 = \frac{1}{6} \frac{\partial}{\partial k^2} D_3(k, m_1^2, \Lambda) \Big|_{k=0} \qquad (10b)$$

gets rid of the divergence in D_3.

But now our mass is divergent

$$m^2 = Z_\phi m_1^2 \approx m_1^2 (1 + g_1^2 z_2)$$

→ redefine m_1^2 to absorb the divergence and take m^2 to be finite

→ renormalized vertex:

$$\Gamma_R^{(2)} = Z_\phi \Gamma^{(2)}$$

This means

$$G^{(2)} = Z_\phi G_R^{(2)}$$

since $G^{(2)}$ is inverse of $\Gamma^{(2)}$

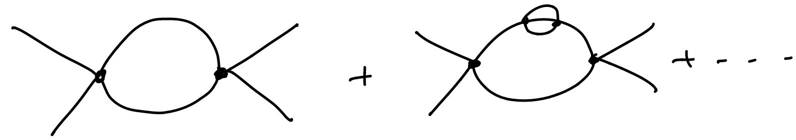
Furthermore, set

$$(*) \quad g = Z_\phi^2 g_1$$

Then, for any graph with E external lines, we have

$$(**) \qquad \Gamma_R^{(E)} = Z_\phi^{E/2} \Gamma^{(E)} \longrightarrow G_{cR}^{(E)} = Z_\phi^{-E/2} G_c^{(4)}$$

For example,



will give rise to

- $6 \; G^{(2)}$'s $\longrightarrow Z_\phi^6$
- 2 vertices $\longrightarrow Z_\phi^{-4}$
- $G_{cR}^{(4)} = Z_\phi^{-2} G_c^{(4)} \longrightarrow$ remaining Z_ϕ^2 factor is canceled!

This is a general story:

recall $\quad I = \frac{1}{2}(nr - E) = 2n - \frac{E}{2} \quad (\phi^4\text{-th})$

$$\longrightarrow Z_\phi^{I+E} = Z_\phi^{2n + \frac{E}{2}} \text{ from } G^{(2)}\text{'s}$$

$(Z_\phi^{-2})^n$ from vertices $\longrightarrow Z_\phi^{E/2}$ is absorbed in $(**)$

Combining now equations (5), (8), (10) with (*) and (**) we see :

$$\Gamma_R^{(2)}(0, m^2, g) = m^2, \qquad (5) + (**)$$

$$\frac{\partial}{\partial K^2} \Gamma_R^{(2)}(K, m^2, g)\Big|_{K^2=0} = 1 \qquad (10)$$

$$\Gamma_R^{(4)}(K_i, m^2, g)\Big|_{K_i=0} = g \qquad (8) + (*) + (**)$$